

STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

ZW 1950-011

On a theorem in the theory of binary relations

J.F. Koksma en H.J.A. Duparc



1950

On a theorem in the theory of binary relations.

by G. Fodor. Szeged.

This paper was inspired by a theorem of Lazar¹⁾ which we shall state below.

Let M be a set of positive measure. To every $x \in M$ we adjoin a set of elements $y \neq x$ of M which may have the cardinal number of the continuum. This means that we define a function $y = \varphi(x)$ of $x \in M$, where $y \in M$, which may assume continuously many values for every x , whereas the equation $x = \varphi(x)$ cannot occur. If neither of the two equations $y = \varphi(x)$ and $x = \varphi(y)$ holds, the two elements x and y are called independent.

Let $\{\varphi(x)\}$ denote the set of values $\varphi(x)$ for a given x ; ^{assume} moreover that x is not a point of accumulation of $\{\varphi(x)\}$ and that $\{\varphi(x)\}$ is of measure zero.

Theorem. We can find a set of positive exterior measure of elements of M that any pair of its elements are independent.

Proof. As the set $\{\varphi(x)\}$ does not contain x and as x is no point of accumulation, its complement $C\{\varphi(x)\}$ with respect to M contains x and an interval I_x surrounding x . Now ~~for every~~ ^{for every} x let us choose in I_x a closed segment $S(x)$ with rational endpoints. Hence all values of $\varphi(x)$ are situated outside $S(x)$.

Now the segments $S(x)$ form an enumerable system S_1, S_2, \dots . To every S_n there belongs at least one $x \in M$, such that $S(x) = S_n$. Let N_n denote the set of all $x \in M$ with $S(x) = S_n$. We assert that there is at least one segment S_n for which N_n has positive exterior measure. For suppose that for every S_n the set N_n would be of measure zero. Then, as $M \subset N_1 + N_2 + \dots$, the well-known theorem that the sum of enumerably many sets of measure zero is a set of measure zero, we must conclude that M was a set of measure zero.

The elements $x \in N_n$ ~~are evidently independent as the adjoined values~~ all belong to S_n ; they are evidently independent as the adjoined values are all outside the segment S_n .

Lazar's theorem²⁾ states only that we can find a set with the power of the continuum under the same assumptions.

¹⁾ Compositio Mathematica 3 (1936), p. 304.

²⁾ Evidently Lazar uses the word "Condensation-point", where only a point of accumulation is meant.